The Buss Reduction for the k-Weighted Vertex Cover Problem

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January 5, 2018

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The 15th International Symposium on Artificial Intelligence and Mathematics (ISAIM 2018)
Fort Lauderdale, Florida, the United States of America

Summary

- For an NP-hard problem, it is desirable to have an algorithm that reduces problem sizes in polynomial time (but does not necessarily solve the problem). This is called a kernelization method.
- The Buss reduction has been known as a kernelization method for the k-vertex cover (k-VC) problem.
- We explicitly generalize it to the k-weighted vertex cover (k-WVC) problem and empirically study its properties.

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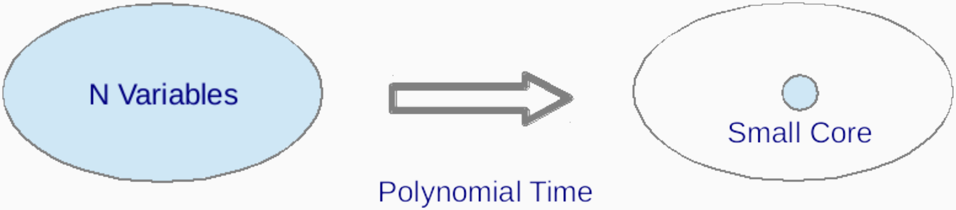
Motivation: the k-Weighted Vertex Cover (k-WVC) Problem

The k-WVC problem: Find a vertex cover with a weight no more than k on a vertex-weighted undirected graph.

Applications:

- Combinatorial auctions (Sandholm 2002)
- Kidney exchange (McCreesh et al. 2017)
- Error correcting code (McCreesh et al. 2017)
- Solving and understanding weighted constraint satisfaction problems (Kumar 2008a, 2016, 2008b)

Motivation: Kernelization and the Buss Reduction



- The k-WVC problem is known to be NP-hard.
- To solve such a problem, an algorithm that reduces the size of the problem in polynomial time is desirable.
- A kernelization method is one such algorithm.
- The Buss reduction is one kernelization method for the k-VC problem.
- Can we generalize the Buss reduction to the k-WVC problem?

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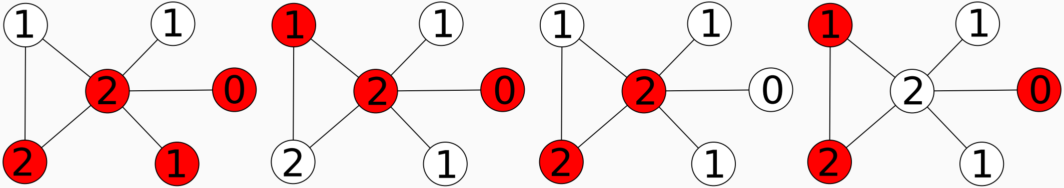
The k-WVC Problem

Given a vertex-weighted undirected graph $G = \langle V, E, w \rangle$,

- A vertex cover is a set $S \subseteq V$ such that every edge in G has at least one endpoint vertex in S.
- The k-WVC problem asks for a vertex cover S with a weight no more than k on G, i.e., $\sum_{v \in S} w(v) \leq k$.
- The k-VC problem is equivalent to the k-WVC problem with all weights equal to 1.

The k-WVC Problem: Example

Example: $(k = 4)$

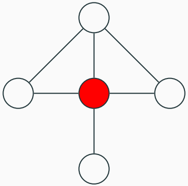


(a) ✗ (b) ✓ (c) ✓ (d) ✗

Red vertices are those vertices in the vertex cover.

The Buss Reduction for the k-VC problem

Intuition: If a vertex has a degree larger than k, it has to be in the vertex cover. Otherwise, all its neighbors have to be in the vertex cover and result in a vertex cover larger than k.



$k = 3$

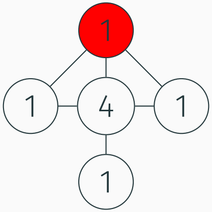
The Buss Reduction for the k-VC problem

The Buss reduction for the k-VC problem on G (Buss et al. 1993):

- Find a vertex v with a degree larger than k and add it to the vertex cover.
- Remove vertex v from G, and the remaining problem is the $(k-1)$-VC problem on the resulting graph.
- Repeat the steps above until $k < 0$ or no vertex can be added to the vertex cover.

The Buss Reduction for the k-WVC problem

Intuition: If a vertex whose neighbors have a total weight larger than k, it has to be in the vertex cover. Otherwise, all its neighbors have to be in the vertex cover and result in a vertex cover with weight larger than k.



$k = 5$

The Buss Reduction for the k-WVC problem

The Buss reduction for the k-WVC problem on G:

- Find a vertex v whose neighbors have a total weight larger than k and add it to the vertex cover.
- Remove vertex v from G, and the remaining problem is the $(k - w(v))$-WVC problem on the resulting graph.
- Repeat the steps above until $k < 0$ or no vertex can be added to the vertex cover.

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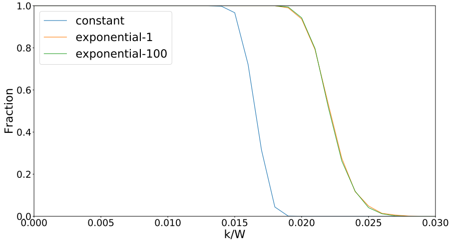
Benchmark Instances

We generated 18 benchmark instance sets with 1,000 benchmark instances each, by using one from each of the following properties:

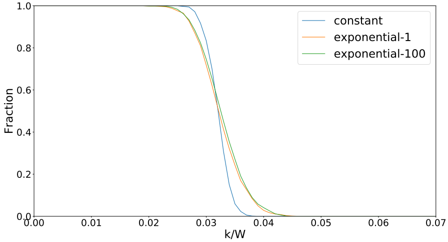
- Random graph model: Erdős-Rényi (ER) and Barabási-Albert (BA)
  - ER: Connectivity $c = 8$
  - BA: $m = m_0 = 2$
- Probabilistic distribution of vertex weights: constant, exponential with $\lambda = 1$ and $\lambda = 100$
  - "Constant distribution" can be somehow viewed as the exponential distribution with $\lambda \to +\infty$ since both of them have zero variance.
  - Note that the exponential distribution with $\lambda = 100$ has a very low variance.
- Number of vertices: 1,000, 500, and 100

No Benchmark Instance Solved

- Experiments showed that no benchmark instance was solved directly using the Buss reduction.
- The Buss reduction typically does not reduce ER or BA graphs to empty kernels if a k-WVC exists.

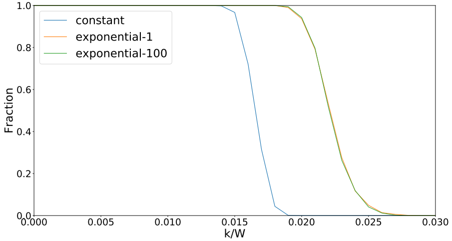


(a) The ER Instances

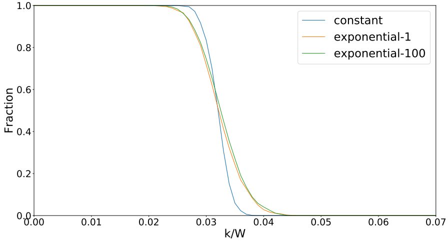


(b) The BA Instances

The fraction of instances η for which the Buss reduction outputs "NO" (no vertex cover with weight not larger than k exists) versus k/W for different weight distributions, where W is the total weight of the vertices in the graph. Only instances with 1,000 vertices are used here. The blue, orange, and green curves represent graphs that have constant, exponential-1, and exponential-100 weights, respectively.



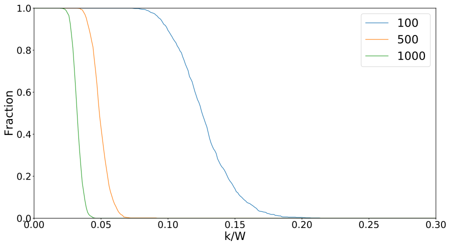
(a) The ER Instances



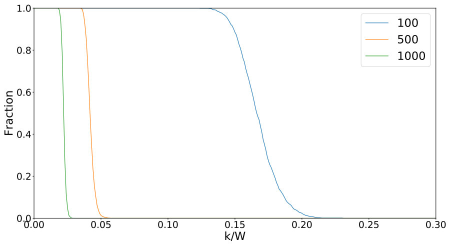
(b) The BA Instances

Observations:

- By changing constant weights to weights sampled by exponential distributions, the critical range (where the phase transition takes place) shifts to larger k's for the ER model and broadens for the BA model.



(a) The ER Instances

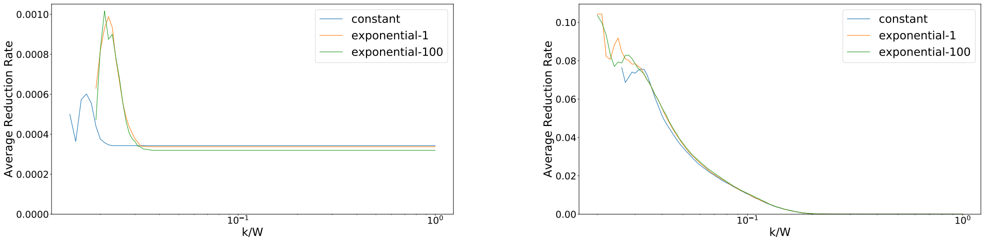


(b) The BA Instances

The fraction of instances η for which the Buss reduction outputs "NO" (no vertex cover with weight not larger than k exists) versus k/W for graphs of different sizes. Only instances with exponential-1 weights are used here.

Observation: As the graph size increases, the critical range narrows and shifts to smaller k's.

Average reduction rate measures how much the problem size has been reduced. (average number of vertices removed divided by number of vertices in the input graph)

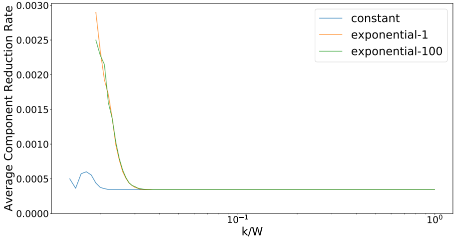


(a) The ER Instances

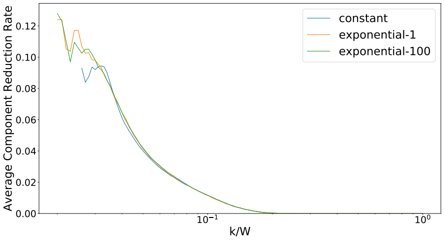
(b) The BA Instances

The average reduction rate as a function of k/W for different weight distributions. Only instances with 1,000 vertices are used here.

Average component reduction rate measures how much the hardness of the problem has been reduced. (one minus the ratio of the number of vertices in the largest connected component of the kernel to the number of vertices in the input graph)



(a) The ER Instances



(b) The BA Instances

Shows the average component reduction rate as a function of k/W for different weight distributions. Only instances with 1,000 vertices are used here.

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Analytical Approximation of Average Reduction Rate

Consider the k-VC problem on a graph with N vertices.

- The Buss reduction starts with $k = k_0$.
- The Buss reduction stops when $k = k_1$.

If we assume that the number of neighbors of vertices does not change from iteration to iteration, then we have

$$1 - F_d(k_1) = \frac{k_0 - k_1}{N},$$

where $F_d(k_1)$ is the fraction of vertices that have degrees less or equal to k_1.

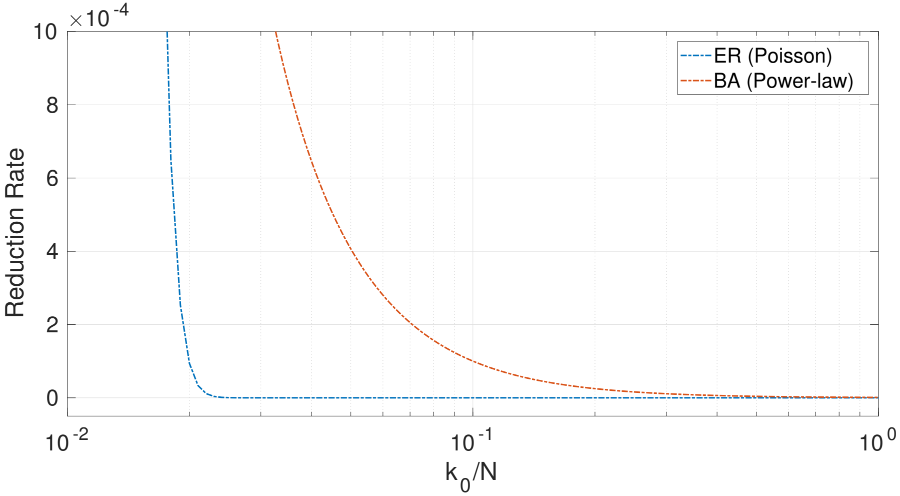
Analytical Approximation of Average Reduction Rate

Similar argument for the k-WVC problem on a graph with N vertices, we have

$$1 - F_\Omega(k_1) = \frac{k_0 - k_1}{N\langle w\rangle},$$

where

- $\langle w\rangle$ is the average vertex weight, and
- $F_\Omega(\cdot)$ is the CDF of $\Omega(v) = \sum_{u\in\partial v} w(u)$.



The plots of the reduction rates as a function of k_0/N obtained by solving $1 - F_d(k_1) = \frac{k_0 - k_1}{N}$, for $N = 1,000$ numerically for ER graphs with parameter $c = 8$ and BA graphs with parameters $m_0 = m = 2$.

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Motivation

The Buss Reduction for the k-Weighted Vertex Cover Problem

Experimental Results

Analysis

Conclusion

Conclusion

- We generalized the Buss reduction to the k-WVC problem.
- We empirically studied it on ER and BA random graphs:
  - The Buss reduction typically does not reduce ER or BA graphs to empty kernels if a k-WVC exists.
  - By changing constant weights to weights sampled by exponential distributions, the critical range shifts to larger k's for the ER model and broadens for the BA model.
  - As the graph size increases, the critical range narrows and shifts to smaller k's.
  - The reduction rate and the component reduction rate drop to near zero quickly for ER instances and more gradually for BA instances.

References I

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