Min-Max Message Passing and Local Consistency in Constraint Networks

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Executive Summary

- Constraint networks (CNs) are important and well known in the constraint programming community.
- Message passing algorithms are important and well known in the probabilistic reasoning community.
- We develop and present the min-max message passing (MMMP) algorithm to connect these two essential concepts.

Agenda

- Constraint Networks (CNs)
- The Min-Max Message Passing (MMMP) Algorithm
- The Modified MMMP Algorithm
- Conclusion

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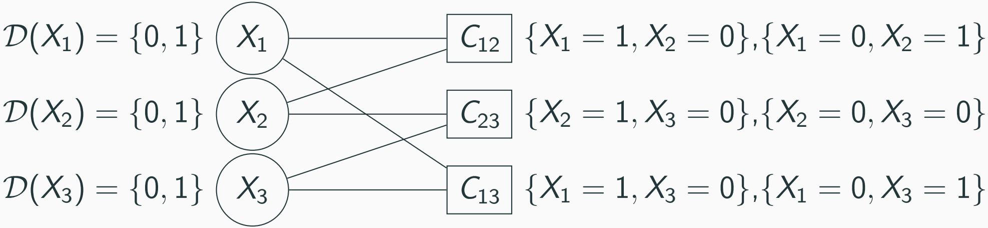
Constraint Networks (CNs)

- A CN is characterized by
  - N discrete-valued variables $\mathcal{X} = \{X_1, X_2, \ldots, X_N\}$
  - Each variable X_i in which has a discrete-valued domain $\mathcal{D}(X_i)$ associated with it.
  - M constraints $\{C_1, C_2, \ldots, C_M\}$
  - Each constraint C_i specifies a list of allowed and disallowed assignments of values to a subset of variables.
  - A solution is an assignment of values to all variables from their respective domains such that all constraints are satisfied.
- It is known to be NP-hard to find a solution (Russell et al. 2009).
- They have been used to solve real-world combinatorial problems, such as map coloring and scheduling (Russell et al. 2009).

Constraint Networks (CNs): Example



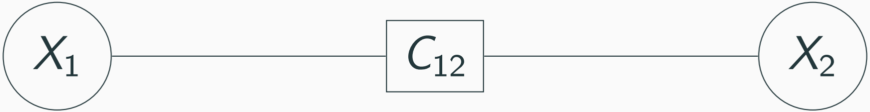
- $\{X_1 = 1, X_2 = 0, X_3 = 0\}$ is a solution, since all constraints are satisfied.
- $\{X_1 = 0, X_2 = 1, X_3 = 0\}$ is not a solution, since C_{13} is violated.

Local Consistency in CNs

- Local consistency of CNs is a class of properties over subsets of variables
- Why is local consistency important?
  - Enforcing local consistency prunes the search space.
  - Enforcing strong k-consistency solves a CN if k is greater than or equal to the treewidth of the CN (Freuder 1982).
  - Enforcing arc consistency is known to solve CNs with only max-closed constraints (Jeavons et al. 1995).

Local Consistency in CNs: Arc Consistency

Is X_1 arc consistent with respect to X_2?



$\mathcal{D}(X_1) = \{0, 1\}$ $\mathcal{D}(X_2) = \{0, 1\}$

- If C_{12} allows $\{X_1 = 0, X_2 = 0\}$ and $\{X_1 = 1, X_2 = 1\}$ ✓
- If C_{12} allows $\{X_1 = 0, X_2 = 0\}$ and $\{X_1 = 0, X_2 = 1\}$ ✗ (No assignment of X_2 is consistent with $\{X_1 = 1\}$)

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The Min-Max Message Passing (MMMP) Algorithm

In a CN, for a constraint C_{ij} over variables X_i and X_j, we define

$$E_{C_{ij}}(X_i = x_i, X_j = x_j) = \begin{cases} 0, & \text{if } C \text{ allows } \{X_i = x_i, X_j = x_j\} \\ 1, & \text{otherwise.} \end{cases}$$

Then minimizing the maximization of all $E_{C_{ij}}$'s produces a solution for the CN!

Based on this idea, the min-max message passing (MMMP) algorithm

- is a variant of belief propagation,
- has information passed locally between variables and constraints via factor graphs,
- has desirable properties (guaranteed convergence) that other message passing algorithms do not have.

Operations on Tables: Min

$$\min_{X_1} \left\{ \begin{array}{|c|c|c|} \hline X_1 \backslash X_2 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ \hline \end{array} \right\} = \begin{array}{|c|c|} \hline X_1 & \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$$

Operations on Tables: Max

max {

X_1 \ X_2	0	1
0	0	1
1	1	0

,

X_1	
0	0
1	1

}

$=$

X_1 \ X_2	0	1
0	$\max\{0,0\}=0$	$\max\{1,0\}=1$
1	$\max\{1,1\}=1$	$\max\{0,1\}=1$

The Min-Max Message Passing (MMMP) Algorithm: Intuition

$$\min_{X_1,X_2,X_3} \max_{X_1,X_2,X_3} \{E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

The Min-Max Message Passing (MMMP) Algorithm: Intuition

$$\min_{X_1,X_2,X_3} \max_{X_1,X_2,X_3} \{E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_2,X_3} \max_{X_2,X_3} \{ \min_{X_1} E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

The Min-Max Message Passing (MMMP) Algorithm: Intuition

$$
\begin{aligned}
&\min_{X_1, X_2, X_3} \max_{X_1, X_2, X_3} \{E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\} \\
&= \min_{X_2, X_3} \max_{X_2, X_3} \{\min_{X_1} E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\} \\
&= \min_{X_2, X_3} \max_{X_2, X_3} \{\nu_{X_2 \to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\}
\end{aligned}
$$

The Min-Max Message Passing (MMMP) Algorithm: Intuition

$$\min_{X_1,X_2,X_3} \max_{X_1,X_2,X_3} \{E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_2,X_3} \max_{X_2,X_3} \{ \min_{X_1} E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_2,X_3} \max_{X_2,X_3} \{\nu_{X_2\to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_3} \max_{X_2} \{\nu_{X_2\to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\}$$

The Min-Max Message Passing (MMMP) Algorithm: Intuition

$$\min_{X_1,X_2,X_3} \max_{X_1,X_2,X_3} \{E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_2,X_3} \max_{X_2,X_3} \{ \min_{X_1} E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_2,X_3} \max_{X_2,X_3} \{\nu_{X_2\to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\}$$

$$= \min_{X_3} \max_{X_2} \{\nu_{X_2\to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\}$$

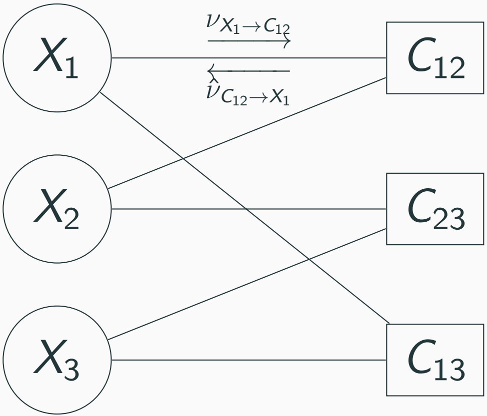
$$= \hat{\nu}_{C_{23}\to X_3}(X_3)$$

The Min-Max Message Passing (MMMP) Algorithm: Intuition

$$
\begin{aligned}
&\min_{X_1,X_2,X_3} \max_{X_1,X_2,X_3} \{E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\} \\
&= \min_{X_2,X_3} \max_{X_2,X_3} \{ \min_{X_1} E_{C_{12}}(X_1, X_2), E_{C_{23}}(X_2, X_3)\} \\
&= \min_{X_2,X_3} \max_{X_2,X_3} \{\nu_{X_2 \to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\} \\
&= \min_{X_3} \max_{X_2} \{\nu_{X_2 \to C_{23}}(X_2), E_{C_{23}}(X_2, X_3)\} \\
&= \hat{\nu}_{C_{23} \to X_3}(X_3)
\end{aligned}
$$

Minimizing $\hat{\nu}_{C_{23} \to X_3}(X_3)$ over X_3 gives the value of X_3 that minimizes the original expression!

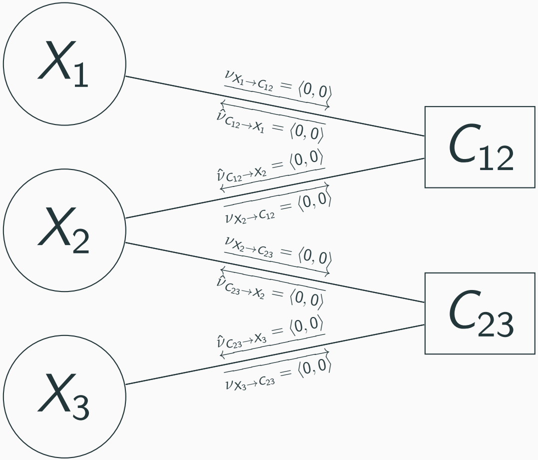
The Min-Max Message Passing (MMMP) Algorithm for CNs



(Xu et al. 2017, Fig. 1)

- A message is a table over the single variable that is common to the sender and the receiver.
- A vertex of k neighbors
  1. applies max on the messages from its $k-1$ neighbors and internal constraint table, and
  2. applies min on the maximization result and sends the resulting table to its k^{th} neighbor.

The MMMP Algorithm: Example



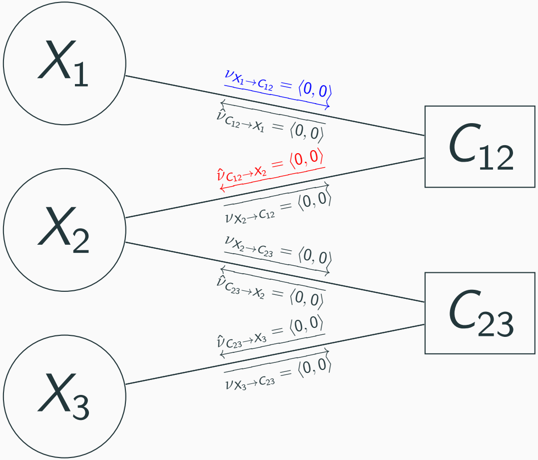
X_1 \ X_2	0	1
0	1	0
1	0	0

(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

The MMMP Algorithm: Example



X_1 \ X_2	0	1
0	1	0
1	0	0

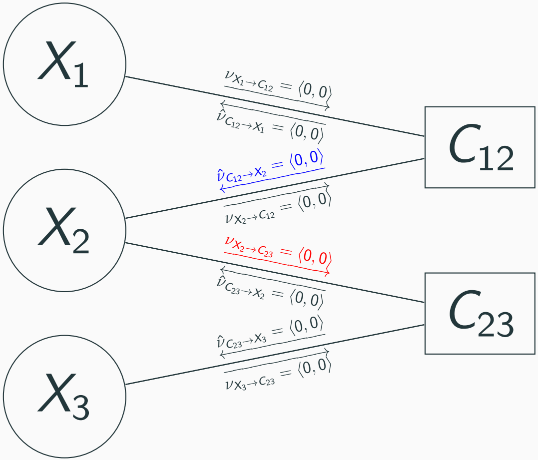
(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

$$\hat{\nu}_{C_{12}\to X_2} = \min_{X_2}\left\{\max\left\{E_{C_{12}}, \nu_{X_1\to C_{12}}\right\}\right\}$$

The MMMP Algorithm: Example



X_1 \ X_2	0	1
0	1	0
1	0	0

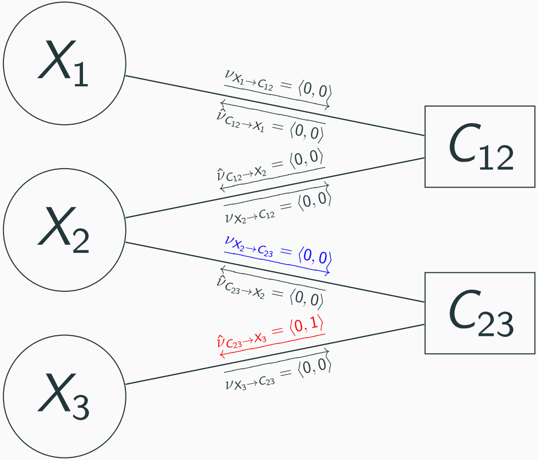
(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

$$\nu_{X_2 \to C_{23}} = \min_{X_2} \{\max \{\hat{\nu}_{C_{12} \to X_2}\}\} = \hat{\nu}_{C_{12} \to X_2}$$

The MMMP Algorithm: Example



X_1 \ X_2	0	1
0	1	0
1	0	0

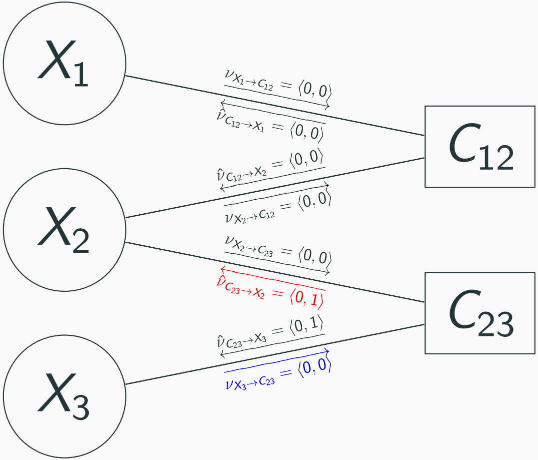
(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

$$\hat{\nu}_{C_{23} \to X_3} = \min_{X_3} \{\max \{E_{C_{23}}, \nu_{X_2 \to C_{23}}\}\}$$

The MMMP Algorithm: Example



X_1 \ X_2	0	1
0	1	0
1	0	0

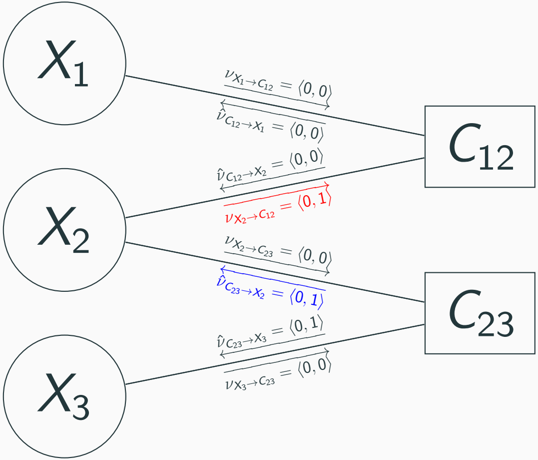
(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

$$\hat{\nu}_{C_{23}\to X_2} = \min_{X_2}\left\{\max\left\{E_{C_{23}}, \nu_{X_3\to C_{23}}\right\}\right\}$$

The MMMP Algorithm: Example



X_1 \ X_2	0	1
0	1	0
1	0	0

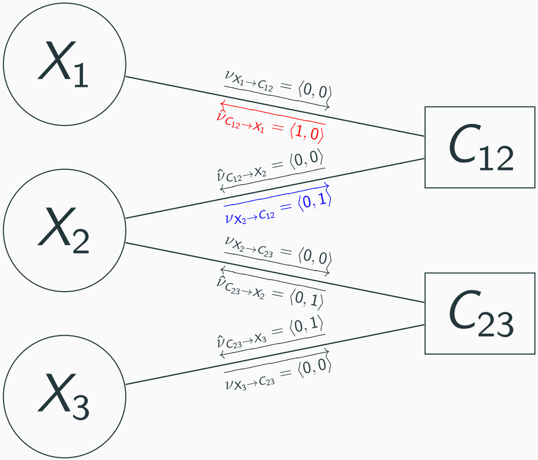
(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

$$\nu_{X_2\to C_{12}} = \min_{X_2}\left\{\max\left\{\hat{\nu}_{C_{23}\to X_2}\right\}\right\} = \hat{\nu}_{C_{23}\to X_2}$$

The MMMP Algorithm: Example



X_1 \ X_2	0	1
0	1	0
1	0	0

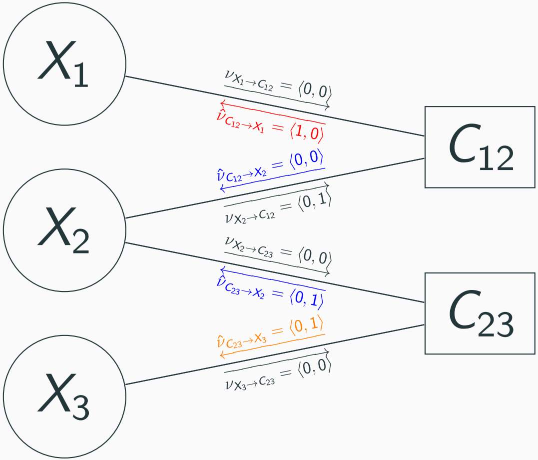
(a) $E_{C_{12}}$

X_2 \ X_3	0	1
0	0	1
1	1	1

(b) $E_{C_{23}}$

$$\hat{\nu}_{C_{12} \to X_1} = \min_{X_1} \left\{ \max \left\{ E_{C_{12}}, \nu_{X_2 \to C_{12}} \right\} \right\}$$

The MMMP Algorithm: Example



- $\max\{\hat{\nu}_{C_{12} \to X_1}(X_1)\} = 0$ iff $X_1 = 1$
- $\max\{\hat{\nu}_{C_{12} \to X_2}(X_2), \hat{\nu}_{C_{23} \to X_2}(X_2)\} = 0$ iff $X_2 = 0$
- $\max\{\hat{\nu}_{C_{23} \to X_3}(X_3)\} = 0$ iff $X_3 = 0$
- solution: $\{X_1 = 1, X_2 = 0, X_3 = 0\}$

Properties of the MMMP Algorithm

- Guaranteed convergence: Unlike other message passing algorithms, the MMMP algorithm guarantees convergence.
- Arc consistency: The solution given by the MMMP algorithm is arc-consistent.
- No solution lost: The solution given by the MMMP algorithm includes all solutions to the CN.

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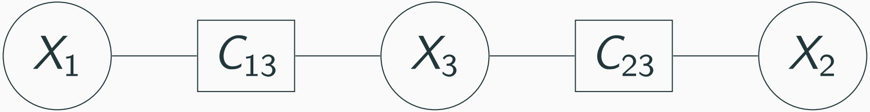
Local Consistency in CNs: Path Consistency

Are X_1 and X_2 path consistent with respect to X_3?

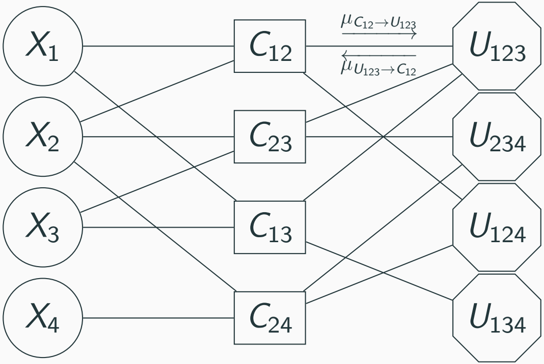


$$\mathcal{D}(X_1) = \{0, 1\} \quad \mathcal{D}(X_3) = \{0, 1\} \quad \mathcal{D}(X_2) = \{0, 1\}$$

- If C_{13} allows $\{X_1 = 0, X_3 = 0\}$ and $\{X_1 = 1, X_3 = 0\}$, C_{23} allows $\{X_2 = 0, X_3 = 0\}$ and $\{X_2 = 1, X_3 = 0\}$ ✓
- If C_{13} allows $\{X_1 = 0, X_3 = 0\}$ and $\{X_1 = 1, X_3 = 1\}$, C_{23} allows $\{X_2 = 0, X_3 = 0\}$ and $\{X_2 = 1, X_3 = 1\}$ ✗ (No assignment of X_3 is consistent with $\{X_1 = 0, X_2 = 1\}$)

Extend the MMMP Algorithm for Path Consistency

The MMMP algorithm can be modified to work on generalized factor graphs to enforce path consistency.



(Xu et al. 2017, Fig. 4)

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Conclusion

- The min-max message passing (MMMP) algorithm is a message passing algorithm that uses the min and max operators.
- The MMMP algorithm connects the message passing techniques with levels of local consistency in constraint networks.
- The MMMP algorithm can be used to enforce arc consistency.
- The MMMP algorithm can be modified to enforce path consistency.
- (Future work) Show the relationship between the MMMP algorithm and k-consistency.

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