Constraint Composite Graph-Based Lifted Message Passing for Distributed Constraint Optimization Problems

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Summary

For solving distributed constraint optimization problems (DCOPs), we develop CCG-Max-Sum, a distributed variant of the lifted min-sum message passing algorithm (Xu et al. 2017) based on the Constraint Composite Graph (Kumar 2008). We experimentally showed that CCG-Max-Sum outperformed other competitors.
Agenda

- Distributed Constraint Optimization Problems (DCOPs)
- The Constraint Composite Graph (CCG)
- CCG-Max-Sum
  - Max-Sum and CCG-Max-Sum
  - The Nemhauser-Trotter (NT) Reduction
- Experimental Evaluation
- Conclusion and Future Work
Agenda

- Distributed Constraint Optimization Problems (DCOPs)
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Cooperative multi-agent system interact to optimize a shared goal. This can be elegantly characterized by DCOPs (Modi et al. 2005; Yeoh et al. 2012).

- Coordination and resource allocation (Léauté et al. 2011; Miller et al. 2012; Zivan et al. 2015)
- Sensor networks (Farinelli et al. 2008)
- Device coordination in smart homes (Fioretto et al. 2017; Rust et al. 2016)
Distributed Constraint Optimization Problems (DCOPs)

- There are $N$ agents $A = \{a_1, a_2, \ldots, a_N\}$, each of which controls one or more variables in $X = \{X_1, X_2, \ldots, X_N\}$, specified by a mapping function $\alpha$. No single variable is controlled by two agents.
- Each variable $X_i$ has a discrete-valued domain $D_i$.
- There are $M$ cost functions (constraints) $F = \{f_1, f_2, \ldots, f_M\}$.
- Each cost function $f_i$ specifies the cost for each assignment $a$ of values to a subset $x_{f_i}$ of the variables (denoted by $f_i(a|x_{f_i})$).
- Find an optimal assignment $a = a^*$ of values to these variables so as to minimize the total cost: $f(a) = \sum_{i=1}^{M} f_i(a|x_{f_i})$.
- Known to be NP-hard.
DCOP Example on Boolean Variables

\[
f(X_1, X_2, X_3) = f_1(X_1) + f_2(X_2) + f_3(X_3) + f_{12}(X_1, X_2) + f_{13}(X_1, X_3) + f_{23}(X_2, X_3)
\]
**DCOP Example: Evaluate the Assignment**

\[ X_1 = 0, X_2 = 0, X_3 = 1 \]

\[
f(X_1 = 0, X_2 = 0, X_3 = 1) = 0.7 + 0.3 + 1.0 + 0.5 + 1.3 + 0.9 = 4.7
\]

(This is not an optimal solution.)
DCOP Example: Evaluate the Assignment $X_1 = 1, X_2 = 0, X_3 = 0$

\[ f(X_1 = 1, X_2 = 0, X_3 = 0) = 0.2 + 0.3 + 0.1 + 0.7 + 0.6 + 0.7 = 2.6 \]

This is an optimal solution. Using brute force, it requires exponential time to find.
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Two Forms of Structure in DCOPs

- Graphical: Which variables are in which cost functions?
- Numerical: How does each cost function relate the variables in it?

How can we exploit both forms of structure computationally?
Minimum Weighted Vertex Cover (MWVC)

Each vertex is associated with a non-negative weight. Sum of the weights on the vertices in the vertex cover is minimized.
Projection of Minimum Weighted Vertex Cover onto an Independent Set

(Kumar 2008, Fig. 2)
Assuming Boolean variables in DCOPs

- Observation: The projection of MWVC onto an independent set looks similar to a cost function.
- Question 1: Can we build the lifted graphical representation for any given cost function? This is answered by (Kumar 2008).
- Question 2: What is the benefit of doing so?
Lifted Representation: Example

\[
f(X_1, X_2, X_3) = f_1(X_1) + f_2(X_2) + f_3(X_3) + f_{12}(X_1, X_2) + f_{13}(X_1, X_3) + f_{23}(X_2, X_3)
\]
Lifted Representations: Example
Constraint Composite Graph (CCG)
An MWVC of the CCG encodes an optimal solution of the original DCOP!
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Max-Sum and CCG-Max-Sum

- Max-Sum (Farinelli et al. 2008; Stranders et al. 2009)
  - is a distributed variant of belief propagation
  - has information passed locally between variables and constraints
- CCG-Max-Sum Algorithm
  - Perform message passing iterations on the MWVC problem instance of the CCG
  - Messages are passed between adjacent vertices
  - Is a distributed variant of the lifted min-sum message passing algorithm (Xu et al. 2017)
- Despite the names, since our goal is to minimize the total cost, all max operators are replaced by min operators.
**Operations on Tables: Min**

\[
\text{min}_{X_1} \left\{ \begin{array}{c|cc}
X_2 & 0 & 1 \\
\hline
X_1 & 0 & 1 & 2 \\
1 & 4 & 3 \\
\end{array} \right\} = \left\{ \begin{array}{c|cc}
X_1 & 0 & 1 \\
\hline
0 & 1 \\
1 & 3 \\
\end{array} \right\}
\]
Operations on Tables: Sum

\[
\begin{array}{c|cc}
X_2 & 0 & 1 \\
\hline
0 & 1 & 2 \\
1 & 4 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cc}
X_1 & 0 & 5 \\
\hline
0 & 5 \\
1 & 6 \\
\end{array}
\]

\[
\begin{array}{c|cc}
X_2 & 0 + 5 = 6 & 2 + 5 = 7 \\
\hline
X_1 & 4 + 6 = 10 & 3 + 6 = 9 \\
\end{array}
\]
Max-Sum

- A message is a table over the single variable, which is the sender or the receiver.
- A vertex of $k$ neighbors
  1. applies sum on the messages from its $k - 1$ neighbors and internal cost function, and
  2. applies min on the summation result and sends the resulting table to its $k^{th}$ neighbor.
Max-Sum

\[ \begin{align*}
\nu_{X_1 \to C_{12}} &= \langle 0, 0 \rangle \\
\nu_{C_{12} \to X_1} &= \langle 0, 0 \rangle \\
\nu_{X_2 \to C_{12}} &= \langle 0, 0 \rangle \\
\nu_{X_2 \to C_{23}} &= \langle 0, 0 \rangle \\
\nu_{C_{23} \to X_2} &= \langle 0, 0 \rangle \\
\nu_{X_3 \to C_{23}} &= \langle 0, 0 \rangle \\
\hat{\nu}_{C_{12} \to X_1} &= \langle 0, 0 \rangle \\
\hat{\nu}_{C_{23} \to X_2} &= \langle 0, 0 \rangle \\
\hat{\nu}_{C_{23} \to X_3} &= \langle 0, 0 \rangle \\
\end{align*} \]

\begin{array}{c|cc}
X_1 & 0 & 1 \\
\hline
0 & 2 & 3 \\
1 & 1 & 2 \\
\end{array} \quad \text{(a) } C_{12}

\begin{array}{c|cc}
X_2 & 0 & 1 \\
\hline
0 & 1 & 4 \\
1 & 2 & 2 \\
\end{array} \quad \text{(b) } C_{23}
Max-Sum

\[ \nu_{X_1 \rightarrow C_{12}} = \langle 0, 0 \rangle \]
\[ \nu_{C_{23} \rightarrow X_3} = \langle 0, 0 \rangle \]
\[ \nu_{X_2 \rightarrow C_{12}} = \langle 0, 0 \rangle \]
\[ \nu_{X_2 \rightarrow C_{23}} = \langle 0, 0 \rangle \]
\[ \nu_{C_{12} \rightarrow X_1} = \langle 0, 0 \rangle \]
\[ \nu_{C_{23} \rightarrow X_2} = \langle 0, 0 \rangle \]

### Table 1: \( C_{12} \)

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

### Table 2: \( C_{23} \)

<table>
<thead>
<tr>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Max-Sum

\[ \nu_{X_1 \rightarrow C_{12}} = \langle 0, 0 \rangle \]
\[ \nu_{C_{12} \rightarrow X_1} = \langle 0, 0 \rangle \]
\[ \nu_{X_2 \rightarrow C_{12}} = \langle 0, 0 \rangle \]
\[ \nu_{X_2 \rightarrow C_{23}} = \langle 0, 1 \rangle \]
\[ \nu_{C_{12} \rightarrow X_2} = \langle 0, 1 \rangle \]
\[ \nu_{C_{23} \rightarrow X_2} = \langle 0, 0 \rangle \]
\[ \nu_{X_3 \rightarrow C_{23}} = \langle 0, 0 \rangle \]
\[ \nu_{C_{23} \rightarrow X_3} = \langle 0, 0 \rangle \]

\[ \begin{array}{ccc}
X_1 & 0 & 1 \\
0 & 2 & 3 \\
1 & 1 & 2 \\
\end{array} \]

(a) \( C_{12} \)

\[ \begin{array}{ccc}
X_2 & 0 & 1 \\
0 & 1 & 4 \\
1 & 2 & 2 \\
\end{array} \]

(b) \( C_{23} \)
Max-Sum

\[ X_1 \quad \nu_{X_1 \rightarrow C_{12}} = (0, 0) \quad \nu_{C_{12} \rightarrow X_1} = (0, 0) \quad \nu_{X_2 \rightarrow C_{12}} = (0, 0) \quad \nu_{X_2 \rightarrow C_{23}} = (0, 1) \quad \nu_{C_{23} \rightarrow X_2} = (0, 0) \quad \nu_{C_{23} \rightarrow X_3} = (0, 2) \quad \nu_{X_3 \rightarrow C_{23}} = (0, 0) \]

\[ C_{12} \]

\[ X_2 \quad \nu_{X_2 \rightarrow C_{12}} = (0, 0) \quad \nu_{C_{12} \rightarrow X_2} = (0, 0) \quad \nu_{X_2 \rightarrow C_{23}} = (0, 1) \quad \nu_{C_{23} \rightarrow X_2} = (0, 0) \]

\[ C_{23} \]

\[ X_3 \quad \nu_{X_3 \rightarrow C_{23}} = (0, 0) \quad \nu_{C_{23} \rightarrow X_3} = (0, 2) \quad \nu_{X_3 \rightarrow C_{23}} = (0, 0) \]

\[ \hat{\nu}_{C_{12} \rightarrow X_1} = (0, 1) \quad \hat{\nu}_{C_{23} \rightarrow X_2} = (0, 0) \quad \hat{\nu}_{C_{23} \rightarrow X_3} = (0, 2) \]

\[ \begin{array}{c|cc|c} \hline X_1 & X_2 & 0 & 1 \\ \hline 0 & 2 & 3 \\ 1 & 1 & 2 \\ \hline \hline \end{array} \]

\( (a) \) \( C_{12} \)

\[ \begin{array}{c|cc|c} \hline X_2 & X_3 & 0 & 1 \\ \hline 0 & 1 & 4 \\ 1 & 2 & 2 \\ \hline \hline \end{array} \]

\( (b) \) \( C_{23} \)
Max-Sum

\[ \begin{array}{c c c}
X_1 & X_2 & X_3 \\
\nu_{X_1 \rightarrow C_{12}} &=& (0,0) \\
\nu_{C_{12} \rightarrow X_1} &=& (0,0) \\
\nu_{X_2 \rightarrow C_{12}} &=& (0,0) \\
\nu_{X_3 \rightarrow C_{12}} &=& (0,1) \\
\nu_{X_2 \rightarrow C_{12}} &=& (0,0) \\
\nu_{X_3 \rightarrow C_{12}} &=& (0,1) \\
\nu_{C_{23} \rightarrow X_2} &=& (0,1) \\
\nu_{C_{23} \rightarrow X_3} &=& (0,2) \\
\nu_{X_3 \rightarrow C_{23}} &=& (0,0) \\
\end{array} \]

\[ \begin{array}{c c c}
X_1 & X_2 & X_3 \\
0 & 2 & 3 \\
1 & 1 & 2 \\
\end{array} \]

\[ \begin{array}{c c c}
X_2 & X_3 & X_1 \\
0 & 1 & 4 \\
1 & 2 & 2 \\
\end{array} \]
Max-Sum

\[ \nu_{X_1 \to C_{12}} = (0, 0) \]
\[ \nu_{C_{12} \to X_1} = (0, 0) \]
\[ \nu_{X_2 \to C_{12}} = (0, 1) \]
\[ \nu_{C_{12} \to X_2} = (0, 1) \]
\[ \nu_{X_3 \to C_{23}} = (0, 1) \]
\[ \nu_{C_{23} \to X_3} = (0, 2) \]
\[ \nu_{X_3 \to C_{23}} = (0, 0) \]

\[
\begin{array}{c|cc}
X_2 & 0 & 1 \\
\hline
X_1 & 0 & 1 \\
0 & 2 & 3 \\
1 & 1 & 2 \\
\end{array}
\]
(a) \( C_{12} \)

\[
\begin{array}{c|cc}
X_3 & 0 & 1 \\
\hline
X_2 & 0 & 1 \\
0 & 1 & 4 \\
1 & 2 & 2 \\
\end{array}
\]
(b) \( C_{23} \)
Max-Sum

\[ \nu_{X_1 \rightarrow C_{12}} = (0, 0), \quad \hat{\nu}_{C_{12} \rightarrow X_1} = (1, 0), \quad \nu_{X_2 \rightarrow C_{12}} = (0, 1), \quad \hat{\nu}_{C_{12} \rightarrow X_2} = (0, 1), \quad \nu_{X_3 \rightarrow C_{23}} = (0, 0), \quad \hat{\nu}_{C_{23} \rightarrow X_3} = \langle 0, 2 \rangle, \quad \nu_{X_3 \rightarrow C_{23}} = \langle 0, 0 \rangle, \quad \hat{\nu}_{C_{23} \rightarrow X_2} = \langle 0, 2 \rangle, \quad \nu_{X_2 \rightarrow C_{23}} = \langle 0, 1 \rangle, \quad \hat{\nu}_{C_{23} \rightarrow X_1} = \langle 1, 0 \rangle. \]

\[
\begin{array}{c|cc}
X_1 & 0 & 1 \\
\hline
0 & 2 & 3 \\
1 & 1 & 2 \\
\end{array}
\quad (a) C_{12}
\]

\[
\begin{array}{c|cc}
X_2 & 0 & 1 \\
\hline
0 & 1 & 4 \\
1 & 2 & 2 \\
\end{array}
\quad (b) C_{23}
\]
Max-Sum

\begin{align*}
\nu_{X_1 \rightarrow C_{12}} &= \langle 0, 0 \rangle \\
\hat{\nu}_{C_{12} \rightarrow X_1} &= \langle 1, 0 \rangle \\
\nu_{X_2 \rightarrow C_{12}} &= \langle 0, 1 \rangle \\
\nu_{X_2 \rightarrow C_{23}} &= \langle 0, 1 \rangle \\
\hat{\nu}_{C_{23} \rightarrow X_2} &= \langle 0, 1 \rangle \\
\nu_{X_3 \rightarrow C_{23}} &= \langle 0, 0 \rangle \\
\nu_{X_3 \rightarrow C_{23}} &= \langle 0, 2 \rangle \\
\hat{\nu}_{C_{23} \rightarrow X_3} &= \langle 0, 2 \rangle
\end{align*}

\begin{itemize}
  \item $X_1 = 1$ minimizes $\hat{\nu}_{C_{12} \rightarrow X_1}(X_1)$
  \item $X_2 = 0$ minimizes $\hat{\nu}_{C_{12} \rightarrow X_2}(X_2) + \hat{\nu}_{C_{23} \rightarrow X_2}(X_2)$
  \item $X_3 = 0$ minimizes $\hat{\nu}_{C_{23} \rightarrow X_3}(X_3)$
  \item Optimal solution: $X_1 = 1, X_2 = 0, X_3 = 0$
\end{itemize}
CCG-Max-Sum: Finding an MWVC on the CCG

• Treat MWVC problems on the CCG as DCOPs and apply Max-Sum on them.

• Messages are simplified passed between adjacent vertices.

\[
\mu^i_{u \rightarrow v} = \max \left\{ W_u - \sum_{t \in N(u) \setminus \{v\}} \mu^{i-1}_{t \rightarrow u}, 0 \right\},
\]
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Motivation: Kernelization and the Nemhauser-Trotter Reduction

- The MWVC problem is known to be NP-hard.
- To solve such a problem, an algorithm that reduces the size of the problem in polynomial time is desirable.
- A kernelization method is one such algorithm.
- The Nemhauser-Trotter (NT) Reduction is one kernelization method for the MWVC problem.
- The Constraint Composite Graph enables the use of the NT reduction.
The Nemhauser-Trotter (NT) Reduction

A is in the minimum weighted VC
B is not in the minimum weighted VC
C and D are in the Kernel
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Experimental Setup

• Algorithms
  • CCG-Max-Sum
  • CCG-Max-Sum-k: CCG-Max-Sum + NT reduction
  • Max-Sum (Farinelli et al. 2008; Stranders et al. 2009)
  • DSA (Zhang et al. 2005)

• Benchmark instances
  • Grid networks (2-d 10 × 10 grids)
  • Scale-free networks (Barabási-Albert model (Barabási et al. 1999)), \( m = m_0 = 2 \)
  • Random networks (Erdős-Rényi model (Erdős et al. 1959)), \( p_1 = 0.4 \) and \( p_1 = 0.8 \), max arity = 4
  • 30 benchmark instances in each instance set, 100 agents/variables
  • Costs are uniformly random numbers from 1 to 100.
Max-Sum CCG-Max-Sum CCG-Max-Sum-k DSA

5,000 iterations for each benchmark instance.

(a) grid networks
(b) scale-free networks
(c) low density random networks ($p_1 = 0.4$)
(d) high-density random networks ($p_1 = 0.8$)
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Conclusion and Future Work

• Conclusion
  • We developed CCG-Max-Sum, a variant of the lifted min-sum message passing algorithm (Xu et al. 2017), for solving DCOPs.
  • We combined NT reduction with CCG-Max-Sum.
  • We experimentally showed the advantage of CCG-Max-Sum.

• Future Work
  • Investigate mixed soft and hard constraints
  • Incorporate Crown reduction (Chlebík et al. 2008)


