# Constraint Composite Graph-Based Lifted Message Passing for Distributed Constraint Optimization Problems

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# Summary

For solving distributed constraint optimization problems (DCOPs), we develop CCG-Max-Sum, a distributed variant of the lifted min-sum message passing algorithm (Xu et al. 2017) based on the Constraint Composite Graph (Kumar 2008). We experimentally showed that CCG-Max-Sum outperformed other competitors.

# Agenda

- Distributed Constraint Optimization Problems (DCOPs)
- The Constraint Composite Graph (CCG)
- CCG-Max-Sum
  - Max-Sum and CCG-Max-Sum
  - The Nemhauser-Trotter (NT) Reduction
- Experimental Evaluation
- Conclusion and Future Work

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# Distributed Constraint Optimization Problems (DCOPs): Motivation

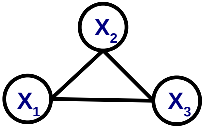
Cooperative multi-agent system interact to optimize a shared goal. This can be elegantly characterized by DCOPs (Modi et al. 2005; Yeoh et al. 2012).

- Coordination and resource allocation (Léauté et al. 2011; Miller et al. 2012; Zivan et al. 2015)
- Sensor networks (Farinelli et al. 2008)
- Device coordination in smart homes (Fioretto et al. 2017; Rust et al. 2016)

# Distributed Constraint Optimization Problems (DCOPs)

- There are $N$ agents $\mathbf{A} = \{a_1, a_2, \ldots, a_N\}$, each of which controls one or more variables in $\mathbf{X} = \{X_1, X_2, \ldots, X_N\}$, specified by a mapping function $\alpha$. No single variable is controlled by two agents.
- Each variable $X_i$ has a discrete-valued domain $D_i$.
- There are $M$ cost functions (constraints) $\mathbf{F} = \{f_1, f_2, \ldots, f_M\}$.
- Each cost function $f_i$ specifies the cost for each assignment $a$ of values to a subset $\mathbf{x}^{f_i}$ of the variables (denoted by $f_i(a|\mathbf{x}^{f_i})$).
- Find an optimal assignment $a = a^*$ of values to these variables so as to minimize the total cost: $f(a) = \sum_{i=1}^{M} f_i(a|\mathbf{x}^{f_i})$.
- Known to be NP-hard.

# DCOP Example on Boolean Variables




| $X_1$ | |
|---|---|
| 0 | 0.7 |
| 1 | 0.2 |

| $X_2$ | |
|---|---|
| 0 | 0.3 |
| 1 | 0.8 |

| $X_3$ | |
|---|---|
| 0 | 0.1 |
| 1 | 1.0 |

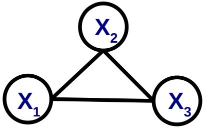
| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 0.5 | 0.6 |
| 1 | 0.7 | 0.3 |

| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.6 | 1.3 |
| 1 | 1.0 | 1.1 |

| $X_1$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.4 | 0.9 |
| 1 | 0.7 | 0.8 |

$$f(X_1, X_2, X_3) = f_1(X_1) + f_2(X_2) + f_3(X_3) + f_{12}(X_1, X_2) + f_{13}(X_1, X_3) + f_{23}(X_2, X_3)$$

# DCOP Example: Evaluate the Assignment $X_1 = 0, X_2 = 0, X_3 = 1$




| $X_1$ | |
|---|---|
| 0 | 0.7 |
| 1 | 0.2 |

| $X_2$ | |
|---|---|
| 0 | 0.3 |
| 1 | 0.8 |

| $X_3$ | |
|---|---|
| 0 | 0.1 |
| 1 | 1.0 |

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 0.5 | 0.6 |
| 1 | 0.7 | 0.3 |

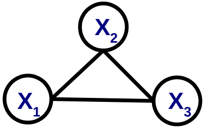
| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.6 | 1.3 |
| 1 | 1.0 | 1.1 |

| $X_1$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.4 | 0.9 |
| 1 | 0.7 | 0.8 |

$$f(X_1 = 0, X_2 = 0, X_3 = 1) = 0.7 + 0.3 + 1.0 + 0.5 + 1.3 + 0.9 = 4.7$$

(This is not an optimal solution.)

# DCOP Example: Evaluate the Assignment $X_1 = 1, X_2 = 0, X_3 = 0$




| $X_1$ | |
|---|---|
| 0 | 0.7 |
| 1 | 0.2 |

| $X_2$ | |
|---|---|
| 0 | 0.3 |
| 1 | 0.8 |

| $X_3$ | |
|---|---|
| 0 | 0.1 |
| 1 | 1.0 |

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 0.5 | 0.6 |
| 1 | 0.7 | 0.3 |

| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.6 | 1.3 |
| 1 | 1.0 | 1.1 |

| $X_1$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.4 | 0.9 |
| 1 | 0.7 | 0.8 |

$$f(X_1 = 1, X_2 = 0, X_3 = 0) = 0.2 + 0.3 + 0.1 + 0.7 + 0.6 + 0.7 = 2.6$$

This is an optimal solution. Using brute force, it requires exponential time to find.

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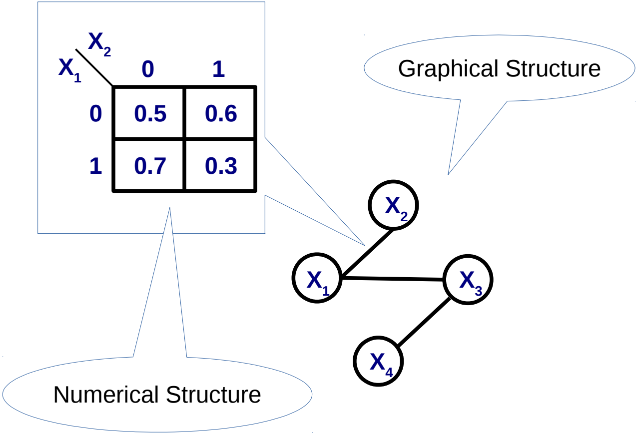
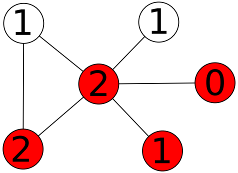
# Two Forms of Structure in DCOPs



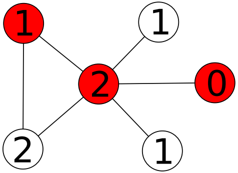

- Graphical: Which variables are in which cost functions?
- Numerical: How does each cost function relate the variables in it?

How can we exploit both forms of structure computationally?

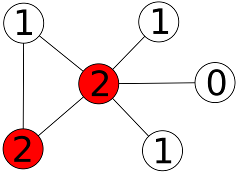
# Minimum Weighted Vertex Cover (MWVC)



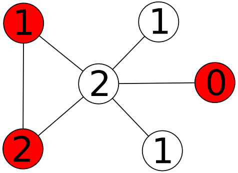

(a) ✗




(b) ✓



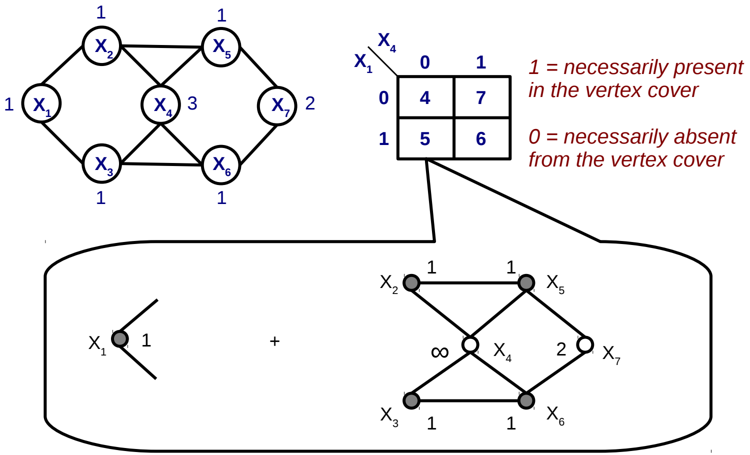

(c) ✗




(d) ✗

Each vertex is associated with a non-negative weight. Sum of the weights on the vertices in the vertex cover is minimized.

# Projection of Minimum Weighted Vertex Cover onto an Independent Set



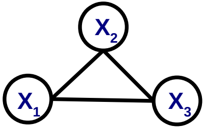

(Kumar 2008, Fig. 2)

# Projection of MWVC onto an Independent Set

Assuming Boolean variables in DCOPs

- Observation: The projection of MWVC onto an independent set looks similar to a cost function.
- Question 1: Can we build the lifted graphical representation for any given cost function? This is answered by (Kumar 2008).
- Question 2: What is the benefit of doing so?

# Lifted Representation: Example

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| 1 | 0.8 |

| $X_3$ | |
|---|---|
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| 1 | 1.0 |

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 0.5 | 0.6 |
| 1 | 0.7 | 0.3 |

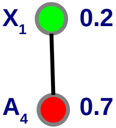
| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.6 | 1.3 |
| 1 | 1.0 | 1.1 |

| $X_1$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.4 | 0.9 |
| 1 | 0.7 | 0.8 |

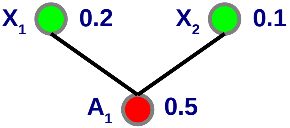
$$f(X_1, X_2, X_3) = f_1(X_1) + f_2(X_2) + f_3(X_3) + f_{12}(X_1, X_2) + f_{13}(X_1, X_3) + f_{23}(X_2, X_3)$$

# Lifted Representations: Example

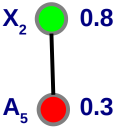
| $X_1$ | |
|---|---|
| 0 | 0.7 |
| 1 | 0.2 |



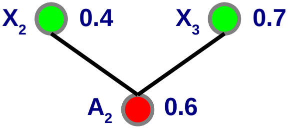

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 0.5 | 0.6 |
| 1 | 0.7 | 0.3 |



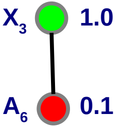

| $X_2$ | |
|---|---|
| 0 | 0.3 |
| 1 | 0.8 |




| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.6 | 1.3 |
| 1 | 1.0 | 1.1 |




| $X_3$ | |
|---|---|
| 0 | 0.1 |
| 1 | 1.0 |




| $X_1$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 0.4 | 0.9 |
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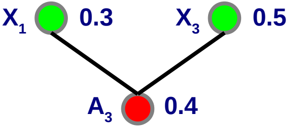
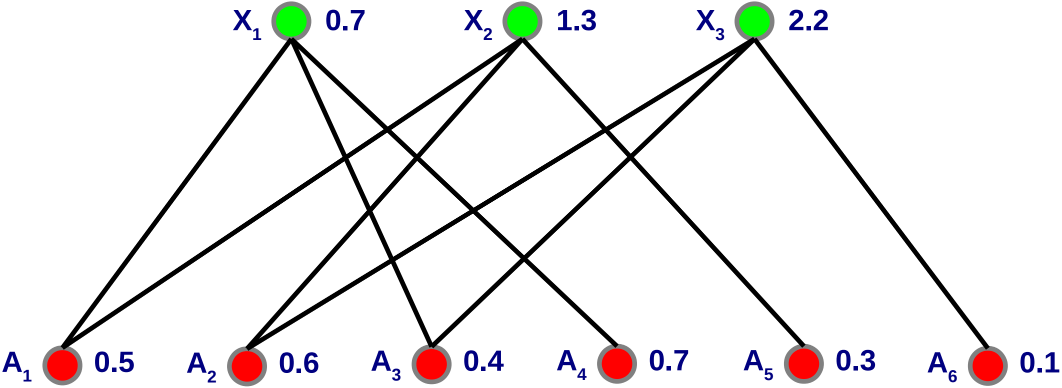
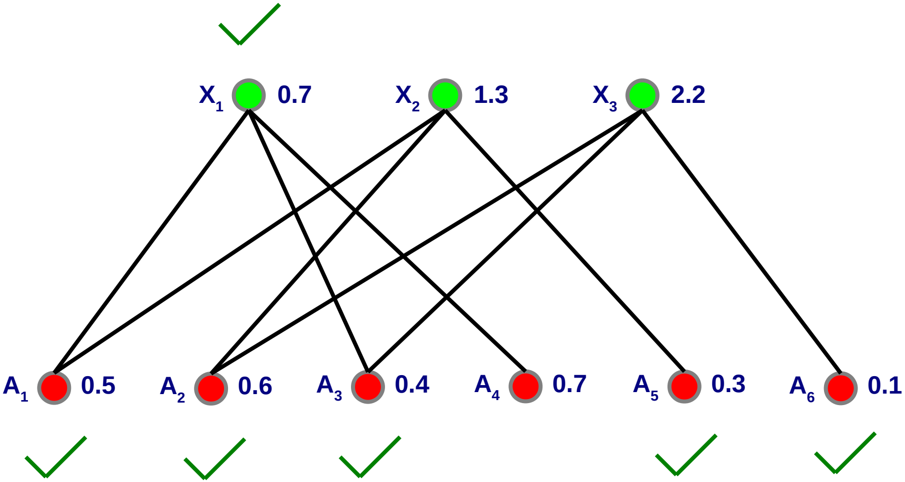


# Constraint Composite Graph (CCG)



# MWVC on the Constraint Composite Graph (CCG)




An MWVC of the CCG encodes an optimal solution of the original DCOP!

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# Max-Sum and CCG-Max-Sum

- Max-Sum (Farinelli et al. 2008; Stranders et al. 2009)
  - is a distributed variant of belief propagation
  - has information passed locally between variables and constraints
- CCG-Max-Sum Algorithm
  - Perform message passing iterations on the MWVC problem instance of the CCG
  - Messages are passed between adjacent vertices
  - Is a distributed variant of the lifted min-sum message passing algorithm (Xu et al. 2017)
- Despite the names, since our goal is to minimize the total cost, all $\max$ operators are replaced by $\min$ operators.

# Operations on Tables: Min

$\min_{X_1}$

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 4 | 3 |

=

| $X_1$ | |
|---|---|
| 0 | 1 |
| 1 | 3 |

# Operations on Tables: Sum

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 4 | 3 |

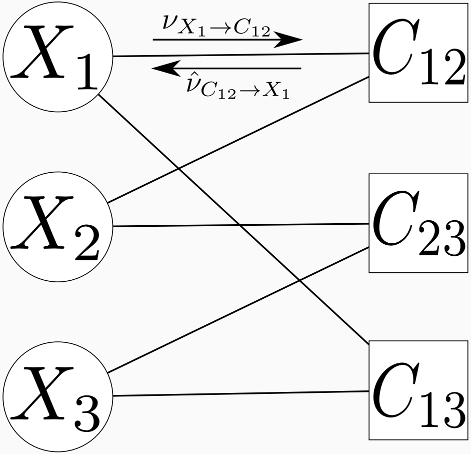
+

| $X_1$ | |
|---|---|
| 0 | 5 |
| 1 | 6 |

=

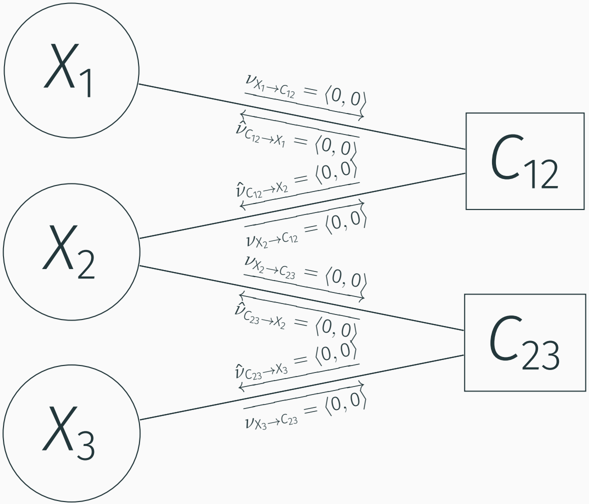
| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | $1 + 5 = 6$ | $2 + 5 = 7$ |
| 1 | $4 + 6 = 10$ | $3 + 6 = 9$ |

# Max-Sum




- A message is a table over the single variable, which is the sender or the receiver.
- A vertex of $k$ neighbors
  1. applies sum on the messages from its $k-1$ neighbors and internal cost function, and
  2. applies min on the summation result and sends the resulting table to its $k^{\text{th}}$ neighbor.

# Max-Sum



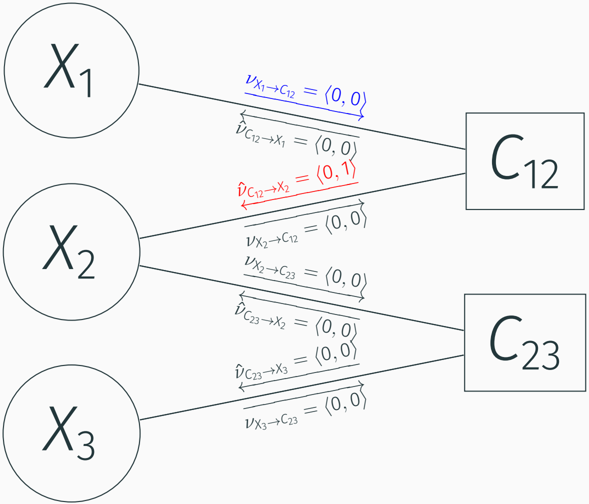

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 1 | 2 |

(a) $C_{12}$

| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 4 |
| 1 | 2 | 2 |

(b) $C_{23}$

# Max-Sum

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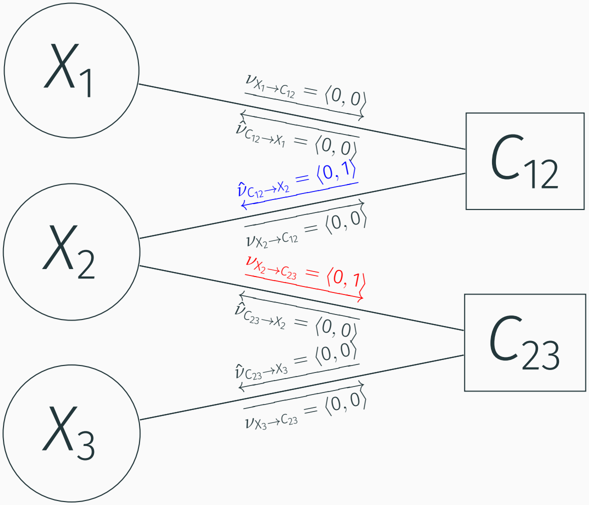


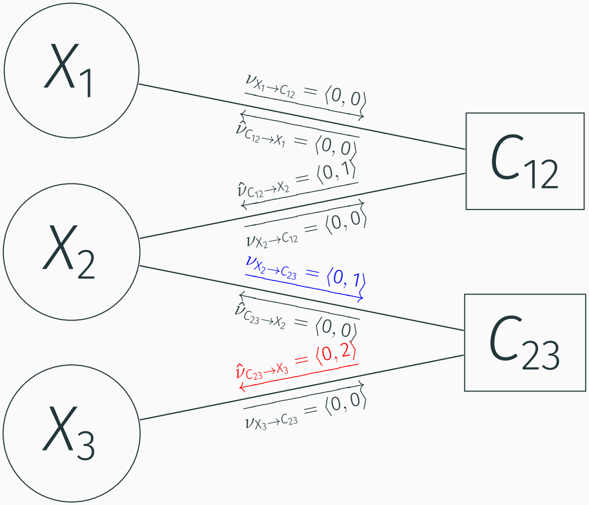

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 1 | 2 |

(a) $C_{12}$

| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 4 |
| 1 | 2 | 2 |

(b) $C_{23}$

# Max-Sum



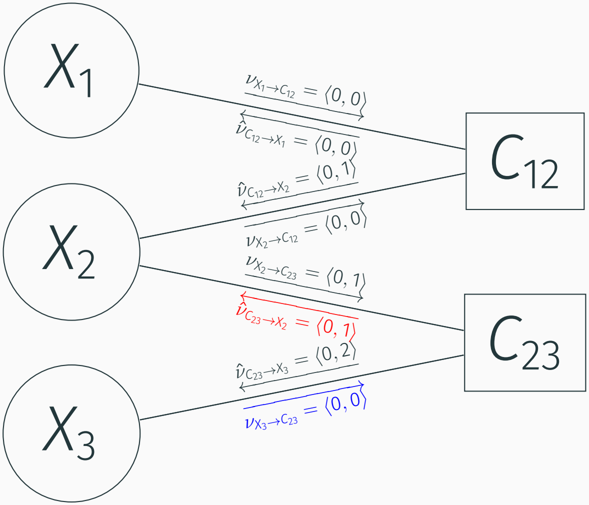

| $X_1 \backslash X_2$ | 0 | 1 |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 1 | 2 |

(a) $C_{12}$

| $X_2 \backslash X_3$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 4 |
| 1 | 2 | 2 |

(b) $C_{23}$

# Max-Sum



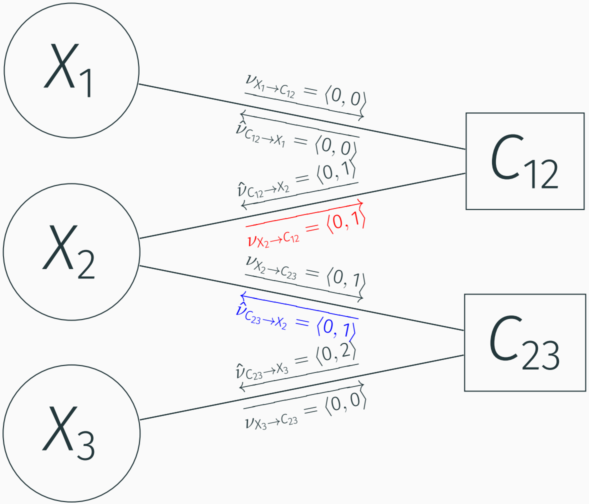

| $X_1$ \ $X_2$ | 0 | 1 |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 1 | 2 |

(a) $C_{12}$

| $X_2$ \ $X_3$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 4 |
| 1 | 2 | 2 |

(b) $C_{23}$

# Max-Sum



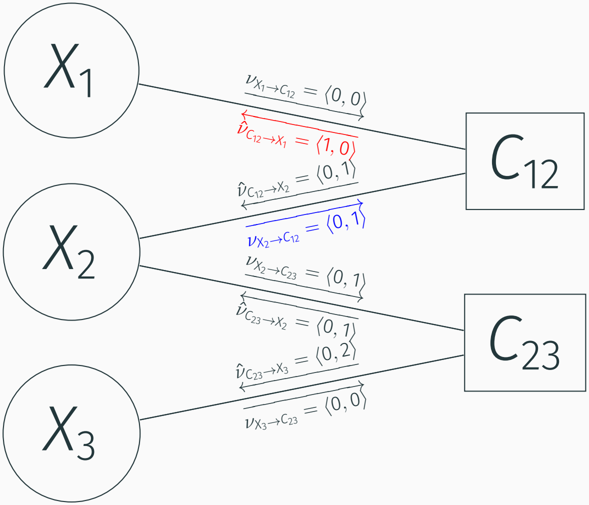

| $X_1 \backslash X_2$ | 0 | 1 |
|---|---|---|
| 0 | 2 | 3 |
| 1 | 1 | 2 |

(a) $C_{12}$

| $X_2 \backslash X_3$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 4 |
| 1 | 2 | 2 |

(b) $C_{23}$

# Max-Sum




| $X_1 \backslash X_2$ | 0 | 1 |
|---|---|---|
| 0 | 2 | 3 |
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(a) $C_{12}$

| $X_2 \backslash X_3$ | 0 | 1 |
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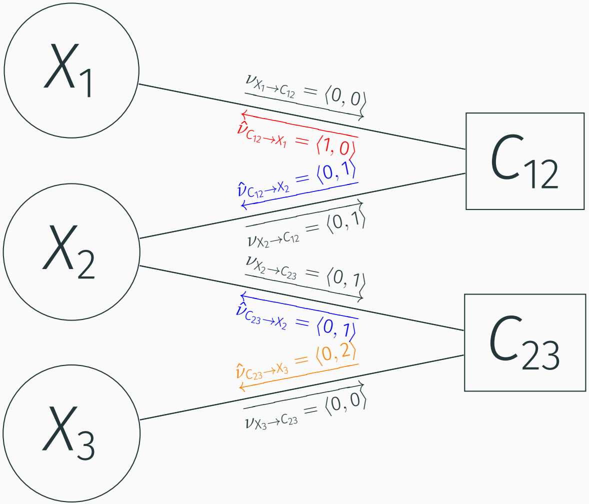
| 0 | 1 | 4 |
| 1 | 2 | 2 |

(b) $C_{23}$

# Max-Sum




- $X_1 = 1$ minimizes $\hat{\nu}_{C_{12}\to X_1}(X_1)$
- $X_2 = 0$ minimizes $\hat{\nu}_{C_{12}\to X_2}(X_2) + \hat{\nu}_{C_{23}\to X_2}(X_2)$
- $X_3 = 0$ minimizes $\hat{\nu}_{C_{23}\to X_3}(X_3)$
- Optimal solution: $X_1 = 1, X_2 = 0, X_3 = 0$

# CCG-Max-Sum: Finding an MWVC on the CCG

- Treat MWVC problems on the CCG as DCOPs and apply Max-Sum on them.
- Messages are simplified passed between adjacent vertices.

$$\mu^{i}_{u\to v} = \max\left\{ w_u - \sum_{t\in N(u)\setminus\{v\}} \mu^{i-1}_{t\to u}, 0 \right\},$$

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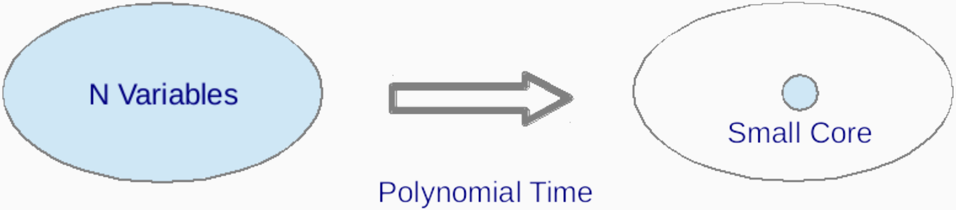
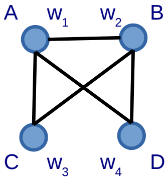
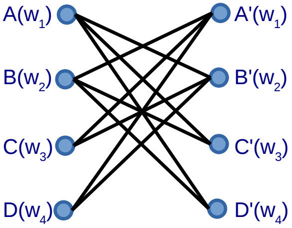
# Motivation: Kernelization and the Nemhauser-Trotter Reduction

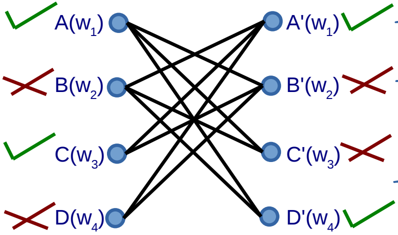



- The MWVC problem is known to be NP-hard.
- To solve such a problem, an algorithm that reduces the size of the problem in polynomial time is desirable.
- A kernelization method is one such algorithm.
- The Nemhauser-Trotter (NT) Reduction is one kernelization method for the MWVC problem.
- The Constraint Composite Graph enables the use of the NT reduction.

# The Nemhauser-Trotter (NT) Reduction




A is in the minimum weighted VC

B is not in the minimum weighted VC

C and D are in the Kernel

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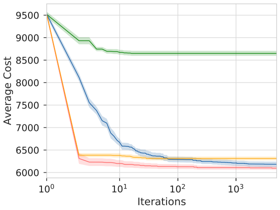
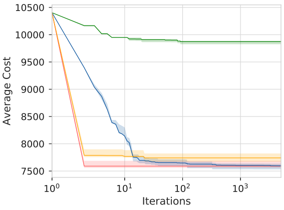
# Experimental Setup

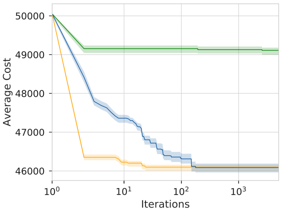
- Algorithms
  - CCG-Max-Sum
  - CCG-Max-Sum-k: CCG-Max-Sum + NT reduction
  - Max-Sum (Farinelli et al. 2008; Stranders et al. 2009)
  - DSA (Zhang et al. 2005)
- Benchmark instances
  - Grid networks (2-d $10 \times 10$ grids)
  - Scale-free networks (Barabási-Albert model (Barabási et al. 1999)), $m = m_0 = 2$
  - Random networks (Erdős-Rényi model (Erdős et al. 1959)), $p_1 = 0.4$ and $p_1 = 0.8$, max arity $= 4$
  - 30 benchmark instances in each instance set, 100 agents/variables
  - Costs are uniformly random numbers from 1 to 100.



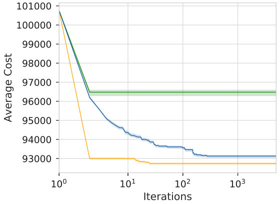

(a) grid networks




(b) scale-free networks




(c) low density random networks ($p_1 = 0.4$)




(d) high-density random networks ($p_1 = 0.8$)

5,000 iterations for each benchmark instance.

# Agenda

- Distributed Constraint Optimization Problems (DCOPs)
- The Constraint Composite Graph (CCG)
- CCG-Max-Sum
  - Max-Sum and CCG-Max-Sum
  - The Nemhauser-Trotter (NT) Reduction
- Experimental Evaluation
- Conclusion and Future Work

# Conclusion and Future Work

- Conclusion
  - We developed CCG-Max-Sum, a variant of the lifted min-sum message passing algorithm (Xu et al. 2017), for solving DCOPs.
  - We combined NT reduction with CCG-Max-Sum.
  - We experimentally showed the advantage of CCG-Max-Sum.
- Future Work
  - Investigate mixed soft and hard constraints
  - Incorporate Crown reduction (Chlebík et al. 2008)

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